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Semiclassical Multiple Orthogonal Polynomials and the Properties of Jacobi–Bessel Polynomials

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This paper deals with Hermite–Padé polynomials in the case where the multiple orthogonality condition is related to semiclassical functionals. The polynomials, introduced in such a way, are a generalization of classical orthogonal polynomials (Jacobi, Laguerre, Hermite, and Bessel polynomials). They satisfy a Rodrigues type formula and an $(s+2)$-order differential equation, where s is the class of the semiclassical functional. A special case of polynomials, multiple orthogonal with respect to the semiclassical weight function $w(x) = x^{\alpha_0}(x-a)^{\alpha_1} e^{\gamma/x}$ (a combination of the classical weights of Jacobi and Bessel), is analyzed in order to obtain the strong (Szegő type) asymptotics and the zero distribution. © 1997 Academic Press

1. INTRODUCTION

In the Introduction we present the basic notions, definitions, and notation of the paper. We sketch the history and the present state of the topic, and discuss the results obtained here.

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1.1. *Classical Orthogonal Polynomials*

Most of the properties which are known for various classical orthogonal polynomials (OP) $\{Q_n(x)\}$

$$\int Q_n(x)\, Q_m(x)\, w(x)\, dx = k_n \delta_{n,m} \tag{1}$$

follow from the fact that their weight functions w satisfy the Pearson differential equation (see [7, 32]),

$$(\phi(z)\, w(z))' + \psi(z)\, w(z) = 0, \tag{2}$$

where

$$\phi \in \mathscr{P}_2, \qquad \psi \in \mathscr{P}_1, \tag{3}$$

and $\mathscr{P}_N$ stands for the set of polynomials of degree not greater than N. The position of the singularities of the differential equation (2) leads to a classification of the different types of classical weights, which are

$$\begin{array}{lll} \text{(J)} & \phi(z) = z(z-a), & w(z) = z^{\alpha_0}(z-a)^{\alpha_1}, \\ \text{(L)} & \phi(z) = z, & w(z) = z^{\alpha}\, e^{\beta z}, \\ \text{(H)} & \phi(z) = const., & w(z) = e^{\beta z^2}, \\ \text{(B)} & \phi(z) = z^2, & w(z) = z^{\alpha} e^{\gamma/z}, \end{array} \tag{4}$$

called Jacobi, Laguerre, Hermite, and Bessel weight functions, respectively. Solutions of the differential equation (2) together with the condition for the path of integration Γ

$$\phi(z)\, w(z)\, z^{\nu}\,|_{\Gamma} = 0, \qquad \nu = 0, 1, 2, ... \tag{5}$$

define an integral moment functional (see [20, 23])

$$\langle \mathbf{w}, z^{\nu} \rangle = w_{\nu} = \int_{\Gamma} z^{\nu} w(z)\, dz. \tag{6}$$

1.2. *Semiclassical Moment Functionals of Class s*

The notion of classical moment functionals (associated with classical weights (4)) is generalized by omitting the restriction (3) on the degrees of the polynomials ϕ, ψ. If

$$s = \max\{\deg \phi - 2, \deg \psi - 1\}, \tag{7}$$

then the functional (6), defined by (2) and (5), is called semiclassical of class s (see [19, 21, 22]). Class $s=0$ corresponds to the classical case, and class $s>0$ to the semiclassical case. During the past decade, the general theory of integral representations of semiclassical functionals has been developed in [13, 21, 22].

1.3. *Semiclassical Orthogonal Polynomials*

The study of orthogonal polynomials with respect to semiclassical functionals (2), (7), (5), and (6), started more than a hundred years ago with the work of Laguerre [17]. In spite of its long history and a number of powerful modern results (see, for example [18, 24, 26]) one cannot say that the theory of semiclassical OP enjoys the same level of development and completeness as the theory of classical polynomials.

The problem is that semiclassical OP do not possess many of the remarkable properties which satisfy classical OP. Among such properties are the existence of Rodrigues type formulae and a simple, exact expression (in terms of polynomials $\phi(z)$ and $\psi(z)$ only) for the coefficients of the differential and recurrence equations.

Also, concerning semiclassical OP, there exists some sort of "classification" problem. As a matter of fact, it has been proven in [4, 21, 22] that the space of solutions of the problem (2), (7), and (5) has dimension $s+1$, which means that for each solution, $w(z)$, of the differential equation (2), there exist $s+1$ homotopically different classes of paths of integration $\{\gamma_j\}_{j=1}^{s+1}$, satisfying (5), such that pairs $\{w(z), \gamma_j\}_{j=1}^{s+1}$ generate, by means of (6), $(s+1)$ linearly independent moment sequences.

Because of the nonlinearity of the construction of OP with respect to the weight, the properties of the OP with respect to the moment functional $\langle \mathbf{w}, p\rangle = \sum_{i=1}^{s+1} \lambda_i \int_{\gamma_i} p(z)\, w(z)\, dz$ cannot be deduced from the properties of the OP with respect to $\langle \mathbf{w}_i, p\rangle = \int_{\gamma_i} p(z)\, w(z)\, dz,\quad i=1, ..., s+1$. The simplest example: let a_1, a_2, and a_3 be three non-colinear points of the complex plane, and $w(z) \equiv const.$ It is comparatively easy to get properties of "canonical" polynomials orthogonal with respect to $w(z)$ placed on the arcs joining two points $\{a_1, a_2\}$ and $\{a_1, a_3\}$. But, properties of the polynomials orthogonal with respect to $w(z)$ placed on the curve, joining three points $\{a_1, a_2, a_3\}$, require more sophisticated analysis (see [11, 26]) and have no connection with the properties of canonical OP.

1.4. *Semiclassical Multiple Orthogonal Polynomials* (*Definition*)

In this paper, to avoid a consideration of $s+1$ sequences of polynomials orthogonal with respect to $s+1$ linearly independent semiclassical functionals $\{w(z), \gamma_j\}_{j=1}^{s+1}$, we introduce a sequence of polynomials which is connected with the entire set of $\{w(z), \gamma_j\}_{j=1}^{s+1}$. To do this, we consider Hermite–Padé (H–P) approximants for the set of functions

$$\hat{w}_{\gamma_j}(z) = \int_{\gamma_j} \frac{w(\xi)}{\xi - z}\, d\xi, \qquad j = 1, 2, ..., s+1. \tag{8}$$

There are several general constructions of H–P approximants (see [3, 25, 27].

In this paper we consider a special case of H–P approximants, called simultaneous rational approximants (see [3]). By definition, a vector of rational functions

$$\left(\frac{P_{n,1}(z)}{Q_{n(s+1)}(z)}, \frac{P_{n,2}(z)}{Q_{n(s+1)}(z)}, ..., \frac{P_{n,s+1}(z)}{Q_{n(s+1)(z)}}\right),$$
$$Q_{n(s+1)} \in \mathscr{P}_{n(s+1)}, \quad P_{n,j} \in \mathscr{P}_n, \quad j = 1, ..., s+1, \tag{9}$$

is called a simultaneous H–P rational approximant of multi-index $(n, ..., n)$ for the vector of functions $(\hat{w}_{\gamma_1}(z), ..., \hat{w}_{\gamma_{s+1}}(z))$ if

$$Q_{n(s+1)}(z)\, \hat{w}_{\gamma_j}(z) - P_{n,j}(z) = O(z^{-n-1}), \qquad |z| \to \infty, \quad j = 1, ..., s+1. \tag{10}$$

The common denominators $Q_{n(s+1)}$ of simultaneous H–P rational approximants (9) possess a property of multiple (or simultaneous) orthogonality (see [3]):

$$\int_{\gamma_j} Q_{n(s+1)}(z)\, z^{\nu} w(z)\, dz = 0, \qquad \nu = 0, ..., n-1, \quad j = 1, 2, ..., s+1. \tag{11}$$

DEFINITION 1.1. The sequence of polynomials $\{Q_{n(s+1)}(z)\}_{n=0}^{\infty}$, $Q_{n(s+1)} \in \mathscr{P}_{n(s+1)}$, contains semiclassical multiple orthogonal polynomials (M-OP) of index $(n, ..., n)$ if they satisfy the $(s+1)$ sets of orthogonality relations (11) with respect to the semiclassical weight function $w(z)$ (2), (7), placed on $(s+1)$ different curves $\{\gamma_j\}_{j=1}^{s+1}$ which satisfy (5) and such that $\langle \mathbf{w}_i, p \rangle = \int_{\gamma_i} p(z)\, w(z)\, dz$ are $s+1$ linearly independent moment functionals.

If $s = 0$, then semiclassical M-OP become classical OP (Jacobi, Laguerre, Hermite, and Bessel).

1.5. *Semiclassical M-OP* (*General Properties and Classification*)

In our paper, we study formal and analytic properties of semiclassical M-OP. Like semiclassical polynomials of usual orthogonality, their partners of multiple orthogonality inherit many remarkable properties of classical orthogonal polyomials.

In Section 2 we prove that semiclassical M-OP satisfy a Rodrigues type formula (Theorem 2.1) and a differential equation of order $(s+2)$ whose coefficients are easily represented by the coefficients of the Pearson equation (2). We consider here the case $s=1$.

In this case, there are seven semiclassical weights, which are classified by the position of the singularities of the Pearson equation (2):

$$\begin{aligned}
&(\text{J–J}) \quad \phi(z)=z(z-a_1)(z-a_2), && w(z)=z^{\alpha_0}(z-a_1)^{\alpha_1}(z-a_2)^{\alpha_2}, && \text{(a)}\\
&(\text{J–B}) \quad \phi(z)=z^2(z-a), && w(z)=z^{\alpha_0}(z-a)^{\alpha_1}e^{\gamma/z}, && \text{(b)}\\
&(\text{B–B}) \quad \phi(z)=z^3, && w(z)=z^{\alpha}\exp\left\{\frac{\gamma_1}{z^2}+\frac{\gamma_2}{z}\right\}, && \text{(c)}\\
&(\text{J–L}) \quad \phi(z)=z(z-a), && w(z)=z^{\alpha_0}(z-a)^{\alpha_1}e^{\beta z} && \text{(d)}\\
&(\text{B–L}) \quad \phi(z)=z^2, && w(z)=z^{\alpha}e^{\beta z}e^{\gamma/z}, && \text{(e)}\\
&(\text{L–H}) \quad \phi(z)=z, && w(z)=z^{\alpha}\exp\{\beta_1 z^2+\beta_2 z\}, && \text{(f)}\\
&(\text{H–H}) \quad \phi(z)=const., && w(z)=\exp\{\beta_1 z^3+\beta_2 z^2+\beta_3 z\}. && \text{(g)}
\end{aligned} \tag{12}$$

Combined with appropriate paths of integration γ_1 and γ_2, satisfying (5), they give rise to linear independent semiclassical functionals (class $s=1$) and corresponding sequences of multiple orthogonal polynomials which, by analogy with the classical case, we call Jacobi–Jacobi (J–J) polynomials, Jacobi–Bessel polynomials (J–B), and so on.

1.6. *Examples of Semiclassical M-OP*

A special case of J–J polynomials ($w(z)=const.$, $\gamma_1=[-1,0]$, $\gamma_2=[0,1]$) was first introduced by Appell [2] in 1901 in connection with a generalization of the Rodrigues formula for Legendre polynomials. Properties of Appell polynomials and some other polynomials generated by a generalized Rodrigues formula have also been investigated in [1, 8, 10, 14]. At the time of the study of these polynomials, their property of multiple orthogonality was not emphasized.

Recently, in connection with the general theory of convergence of H–P approximants and asymptotics of H–P polynomials, some results concerning M-OP with weights from (12) have been obtained (see [9, 27]). In 1979, Kalyagin (see [15]) proved formulae of strong (Szegő type) asymptotics for the special case of J–J polynomials: $w(x)$ as in (12(a)) with $a_1=-a_2=a$, $\gamma_1=[-a,0]$, $\gamma_2=[0,a]$ (see also [16]). Sorokin studied formal and analytic properties (including strong asymptotics) for J–L (case $a<0$, $\beta<0$), L–H and H–H polynomials (see [28–31]).

1.7. *Asymptotics and Zero Distribution of Jacobi–Bessel Polynomials*

In Section 3, we study the J–B polynomials in more detail. This type of semiclassical M-OP has not been studied before.

These polynomials $\{Q_{2n}\}$, $Q_{2n} \in \mathscr{P}_{2n}$ are multiple orthogonal (11) with respect to the weight function (see 12(b))

$$w(z) = z^{\alpha_0}(z-a)^{\alpha_1}\, e^{\gamma/z}, \qquad \Re\alpha_1 > -1, \qquad \gamma \neq 0, \tag{13}$$

taken on the two canonical paths, which are (see Fig. 1) the following:

- γ_1 is a Jordan arc, leaving the origin from the side of the half-plane $\{z: \Re(\gamma/z) < 0\}$ and ending at the point a.
- γ_2 is a Jordan curve, leaving the zero point from the side of half-plane $\{z: \Re(\gamma/z) < 0\}$, turning around zero and ending at the zero point again from the side of the half-plane $\{z: \Re(\gamma/z) < 0\}$.

If $a > 0$, and $\gamma < 0$, then γ_1 becomes the interval $[0, a]$ and γ_2 the circumference around zero connected with the origin by the cut along the segment of the positive semiaxis. (If α_0 and α_1 are integers then γ_2 is only the circumference around zero.)

So, the polynomial Q_{2n}, $\deg Q_{2n} = 2n$, with respect to the weight (13) satisfies n orthogonality relations along the curve γ_1 (like Jacobi polynomials)

$$\int_{\gamma_1} Q_{2n}(z)\, z^{\nu} w(z)\, dz = 0, \qquad \nu = 0, 1, ..., n-1,$$

and n other orthogonality relations along γ_2 (like Bessel polynomials)

$$\int_{\gamma_2} Q_{2n}(z)\, z^{\nu} w(z)\, dz = 0, \qquad \nu = 0, 1, ..., n-1.$$

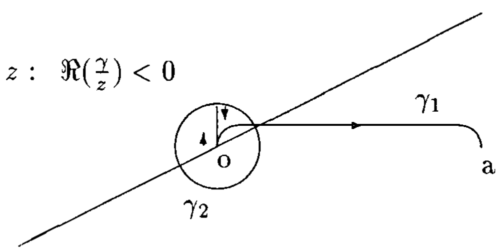

FIGURE 1

The aim of Section 3 is to describe the asymptotic behaviour of the polynomials $\{Q_{2n}\}$, when $n \to \infty$, and also their zero distribution. We start with the representation of the generating function for J–B polynomials:

$$F(x, z) = \sum_{n=0}^{\infty} \frac{Q_{2n}(x)}{n!} \frac{1}{z^{n+1}}. \tag{14}$$

In order to investigate the asymptotics of the polynomials $\{Q_{2n}\}$, we use the Darboux method (see [25, 32]). This method is based on the study of the singularities in the complex z-plane for the generating function $F(x, z)$. The behaviour of the singularities on the boundary of the largest disk of holomorphy of $F(x, z)$ describes the limit of the coefficients of the power expansion (14), i.e., asymptotics of the polynomials $\{Q_{2n}\}$ when $n \to \infty$. We summarize the investigation of the generating function and its singular points in Theorems 3.1 and 3.5. Then, we deduce, from these theorems the asymptotic behaviour of the polynomials $\{Q_{2n}\}$ in the whole complex plane (Theorem 3.4).

Finally, we apply the formula obtained for the strong (Szegő type) asymptotics of the J–B polynomials to the investigation of their zero distribution. A conclusion that can be drawn from the investigation is that about half of the zeros of the polynomial $Q_{2n}(x)$ (for n large enough) are concentrated on the interval $[\frac{1}{9}a, a]$ (see Theorem 3.5); the rest of the zeros tend to the origin when $n \to \infty$.

Moreover, if we denote $Q_{2n} = k_n \prod_{\nu=1}^{2n} (x - x_{\nu, 2n}) = k_n q_{n,1}(x)\, q_{n,2}(x)$, where $q_{n,1} = \prod_{\nu=1}^{N_n} (x - x_{\nu, 2n})$ is the part of the polynomials corresponding to the zeros concentrated on the interval $[\frac{1}{9}a, a]$, then the zero counting measure of the polynomials $q_{n,1}$,

$$\nu_n := \nu[q_{n,1}] := \frac{1}{n} \sum_{\nu=1}^{N_n} \delta(x - x_{\nu, 2n}), \tag{15}$$

has a weak limit $(n \to \infty)$

$$\nu_n \xrightarrow{*} \nu.$$

Here ν is the positive probability measure on $[0, a]$, which is the equilibrium measure with respect to the logarithmic potential in the presence of the external field generated by a unit charge placed at the origin.

In comparison with Jacobi polynomials $(w(z) = z^{\alpha_0}(z-a)^{\alpha_1}, \gamma = [0, a])$ whose zeros are dense on the interval $[0, a]$ and asymptotically distributed according to the equilibrium measure from logarithmic potential theory (without any external field), the limit of the zero counting measure (15) ν has support $[a/9, a]$. This means that the part $q_{n,2}$ of the polynomials Q_{2n}, whose zeros tend to the origin (we could call it the "Bessel part" of the J–B polynomial), creates an external field which pushes the zeros of $q_{n,1}$ (the "Jacobi part" of the J–B polynomial) out of the subinterval $[0, a/9] \subset [0, a]$ (their natural home place). Such an effect, called "push of the zeros," is rather common for the general Hermite–Padé polynomials (see [12, 27]).

1.8. *Conclusion*

Summarizing, we would like to mention that the notion of multiple orthogonality (in comparison with "usual" orthogonality) seems to be very appropriate for the generalization of classical OP. The semiclassical M-OP are easily classified; they reflect the multidimensionality features of the set of semiclassical moment functionals of class s. The semiclassical M-OP inherit most of the remarkable properties of classical OP. As a consequence, the powerful tools which have been created for the treatment of classical OP may be used (after adaptation) for an analysis of semiclassical M-OP. This gives the opportunity for the theory of semiclassical M-OP to develop as far as the theory of classical OP. At the same time, the semiclassical M-OP show new interesting phenomena which have not occurred in the theory of classical OP. They present a new set of special functions which is very important in several applications.

2. DIFFERENTIAL PROPERTIES OF SEMICLASSICAL MULTIPLE ORTHOGONAL POLYNOMIALS OF CLASS $s=1$

In this section, we prove the Rodrigues type formula and derive a differential equation for general semiclassical M-OP of class $s=1$. As mentioned in the Introduction, the polynomials $Q_{2n} \in \mathscr{P}_{2n}$ satisfy the multiple orthogonal relations

$$\begin{cases} \int_{\gamma_1} Q_{2n}(z)\, z^{\nu} w(z)\, dz = 0 \\ \int_{\gamma_2} Q_{2n}(z)\, z^{\nu} w(z)\, dz = 0 \end{cases} \qquad \nu = 0, 1, ..., n-1, \tag{16}$$

with respect to the canonical integral representation $\{w(z), \gamma_1, \gamma_2\}$ of the semiclassical functional of the class $s=1$, i.e., (see (2), (7), and (5)),

$$\begin{aligned} &\text{(i)} \quad (\phi(z)\, w(z))' + \psi(z)\, w(z) = 0 \\ &\text{(ii)} \quad \max\{\deg[\phi] - 2, \deg[\psi] - 1\} = 1 \\ &\text{(iii)} \quad \phi(z)\, w(z)\, z^{\nu}\big|_{\gamma_{1,2}} = 0, \qquad \nu = 0, 1, 2, ... \end{aligned} \tag{17}$$

2.1. *Rodrigues Type Formulas*

THEOREM 2.1. *Let* $\{Q_{2n}\}_{n=0}^{\infty}$ *be a sequence of semiclassical M-OP (class* $s=1$*) with index* (n, n) *(i.e.,* (16), *and* (17) *hold) Then*

$$Q_{2n}(z)=\frac{1}{w(z)}\, D^n[\phi^n(z)\, w(z)], \tag{18}$$

where $D=d/dz$.

Proof. First, we prove that the expression on the righ-hand side of (18) is a polynomial in $\mathscr{P}_{2n}$. It is easy to check by induction that $(1/w)\, D^k[\phi^n w]$ is a polynomial for any $k=0, 1, 2, ..., n$. We have from (17i) for $k=1$,

$$D[\phi^n w]=D[(\phi w)\, \phi^{n-1}]=[-\psi+(n-1)\, \phi']\, \phi^{n-1}w=: b_{n,1}\phi^{n-1}w,$$

where

$$b_{n,1}(z)=-\psi(z)+(n-1)\, \phi'(z)$$

is a polynomial of degree at most 2 (see (17ii)). Introducing recursively a sequence of polynomials

$$b_{n,k}(z)=b'_{n,k-1}(z)\, \phi(z)+b_{n,k-1}(z)\, b_{n-k+1,1}(z), \qquad k=2, 3, ..., n$$

($b_{n,k}\in\mathscr{P}_{2k}$), we have

$$D^k[\phi^n w]=b_{n,k}\phi^{n-k}w. \tag{19}$$

Indeed, using induction, we have

$$D^k[\phi^n w]=D[(b_{n,k-1}w\phi)\, \phi^{n-k}]=(b'_{n,k-1}\phi+b_{n,k-1}b_{n-k+1,1})\, \phi^{n-k}w.$$

So, the polynomials on the right-hand of (18) belong to $\mathscr{P}_{2n}$.

Second, we prove that the polynomials on the right-hand side of (18) satisfy multiple orthogonal relations (16). For $j=1, 2$ and $0\leqslant v\leqslant n-1$ we have

$$\int_{\gamma_j} D^n[\phi^n(z)\, w(z)]\, z^v\, dz$$

$$=z^v\, D^{n-1}[\phi^n(z)\, w(z)]|_{\gamma_j}-v\int_{\gamma_j} D^{n-1}[\phi^n(z)\, w(z)]\, z^{v-1}\, dz.$$

Taking into account (19) and (17iii), we continue integrating by parts $(v-1)$ times

$$\begin{aligned}\int_{\gamma_j} D^n[\phi^n(z)\, w(z)]\, z^v\, dz &= (-1)^v\, v! \int_{\gamma_j} D^{n-v}[\phi^n(z)\, w(z)]\, dz \\ &= (-1)^v\, v!\, D^{n-v-1}[\phi^n w]|_{\gamma_j} \\ &= (-1)^v\, v!\, b_{n,\, n-(v+1)}(z)\, \phi^{v+1}(z)\, w(z)|_{\gamma_j} = 0\end{aligned}$$

2.2. *The Differential Equation of Third Order for Semiclassical M-OP of Class $s=1$*

THEOREM 2.2. *Let $\{Q_{2n}\}_{n=0}^{\infty}$ be a sequence of semiclassical M-OP (class $s=1$) with index (n, n) (i.e., (16) and (17) hold). Then*

$$\phi^2(z)\, Q'''_{2n}(z) - 2\psi(z)\, \phi(z)\, Q''_{2n}(z) + A_1(z; n)\, Q'_{2n}(z) + A_2(z; n)\, Q_{2n}(z) = 0, \tag{20}$$

where

$$\begin{aligned}A_1(z; n) &= \psi(\psi + \phi') - \phi\left(\frac{n(n-1)}{2}\, \phi'' - (n-1)\, \psi'\right), \\ A_2(z; n) &= (\psi + \phi')\left(\frac{n(n-1)}{2}\, \phi'' - n\psi'\right) \\ &\quad - \phi\left(\frac{n(n-1)(2n+5)}{6}\, \phi''' - \frac{n(n+3)}{2}\, \psi''\right).\end{aligned} \tag{21}$$

Proof. We introduce the function

$$y_n = D^n[\phi^n w].$$

Applying Leibnitz's rule in the expression $D^{n+3}[\phi\phi^n w]$ we first obtain

$$\begin{aligned}D^{n+3}[\phi^{n+1} w] &= \phi y'''_n + (n+3)\, \phi' y''_n\, \frac{(n+3)(n+2)}{2}\, \phi'' y'_n \\ &\quad + \phi''' \frac{(n+3)(n+2)(n+1)}{6}\, y_n.\end{aligned} \tag{22}$$

On the other hand,

$$D^{n+3}[\phi^{n+1} w] = D^{n+2}[n\phi^{n-1}\phi'(\phi w) - \phi^n \psi w] = D^{n+2}[(n\phi' - \psi)\, \phi^n w].$$

Again, using Leibnitz's rule, we deduce that the expression above is equal to

$$(n\phi' - \psi)\, y_n'' + (n+2)(n\phi'' - \psi')\, y_n' + \frac{(n+2)(n+1)}{2}(n\phi''' - \psi'')\, y_n. \tag{23}$$

From (22) and (23) we have

$$\phi y_n''' + (3\phi' + \psi)\, y_n'' + (n+2)\left\{\frac{3-n}{2}\phi'' + \psi'\right\} y_n' + \frac{(n+2)(n+1)}{6}\{(3-2n)\,\phi''' + 3\psi''\}\; y_n = 0. \tag{24}$$

But $y_n = wQ_{2n}$, and substitution of y_n', y_n'', and y_n''' in (24) yields

$$\begin{aligned} &\phi w Q_{2n}''' + \{3\phi w' + (3\phi' + \psi)\, w\}\; Q_{2n}'' \\ &+ \left\{3\phi w'' + 2(3\phi' + \psi)\, w' + (n+2)\left(\frac{3-n}{2}\phi'' + \psi'\right) w\right\} Q_{2n}' \\ &+ \left\{\phi w''' + (3\phi' + \psi)\, w'' + (n+2)\left(\frac{3-n}{2}\phi'' + \psi'\right) w' \right. \\ &\left. + \frac{(n+2)(n+1)}{6}((3-2n)\,\phi''' + 3\psi'')\, w\right\} Q_{2n} = 0. \end{aligned} \tag{25}$$

In order to eliminate the dependence on derivatives of $w(z)$ from the coefficients of (25), we apply the Pearson equation

$$\phi\frac{w'}{w} + \psi + \phi' = 0. \tag{26}$$

From $D[\phi w] + \psi w = 0$ follows

$$D^2[\phi w] + \psi' w + \psi w' = 0,$$

but, using Leibnitz's rule, we have

$$D^2[\phi w] = \phi'' w + 2\phi' w' + \phi w'',$$

which gives us

$$\phi\frac{w''}{w} + (\psi + 2\phi')\frac{w'}{w} + (\psi' + \phi'') = 0. \tag{27}$$

By the same method, we obtain

$$\phi\,\frac{w'''}{w}+(\psi+3\phi')\,\frac{w''}{w}+(2\psi'+3\phi'')\,\frac{w'}{w}+(\psi''+\phi''')=0. \tag{28}$$

Substituting the relations (26)–(28) into Eq. (25), we have

$$\begin{aligned}
&\phi Q_{2n}'''-\{3(\psi+\phi')-(3\phi'+\psi)\}\;Q_{2n}''\\
&\quad-\left\{3(\psi+2\phi')\,\frac{w'}{w}+3(\psi'+\phi'')-2(3\phi'+\psi)\,\frac{w'}{w}\right.\\
&\quad\left.-(n+2)\left(\frac{3-n}{2}\,\phi''+\psi'\right)\right\}\;Q_{2n}'\\
&\quad-\left\{(2\psi'+3\phi'')\,\frac{w'}{w}+(\psi''+\phi''')-(n+2)\left(\frac{3-n}{2}\,\phi''+\psi'\right)\frac{w'}{w}\right.\\
&\quad\left.-\frac{(n+2)(n+1)}{6}\,((3-2n)\,\phi'''+3\psi'')\right\}\;Q_{2n}=0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\phi Q_{2n}'''-2\psi Q_{2n}''-\left\{\psi\,\frac{w'}{w}+\frac{n(n-1)}{2}\,\phi''-(n-1)\,\psi'\right\}\;Q_{2n}'\\
&\quad+\left\{\left(\frac{n(1-n)}{2}\,\phi''+n\psi'\right)\frac{w'}{w}+\frac{n(n+3)}{2}\,\psi''-\frac{n(n-1)(2n+5)}{6}\,\phi'''\right\}\;Q_{2n}=0.
\end{aligned}$$

Multiplying the above equation by ϕ and using the Pearson equation again, we obtain (20) and (21). ∎

The case $(w(z)\equiv 1,\ \phi(z)=z(z-1)(z-a)$ has been studied in [15]. In this case $w'/w=0$ and

$$\begin{aligned}
\phi'(z)&=-\psi(z)=3z^2-2(a+1)\,z+a\\
\phi''(z)&=-\psi'(z)=6z-2(a+1)\\
\phi'''(z)&=-\psi''(z)=6.
\end{aligned}$$

If we substitute these expressions in (20) and (21), we have

$$\phi Q_{2n}'''-2\psi Q_{2n}''+\frac{(n+2)(n-1)}{2}\,\psi' Q_{2n}'-2n(n+1)(n+2)\;Q_{2n}=0,$$

which is the same result as in [16, Theorem 1].

3. ANALYTIC PROPERTIES OF JACOBI–BESSEL POLYNOMIALS

In this section we derive the strong asymptotics for J–B polynomials and then apply this to the study of their zero distribution. As mentioned in the Introduction, by J–B polynomials we mean the sequence of polynomials $\{Q_{2n}\}$ which are multiple orthogonal (16) with respect to the weight function

$$w(x) = x^{\alpha_0}(x-a)^{\alpha_1} e^{\gamma/x}, \qquad \Re\alpha_1 > -1, \quad \gamma \neq 0 \tag{29}$$

(a, α_0, α_1, and γ can be complex unless stated otherwise stated), placed on the two canonical contours $\gamma_1 = [0, a]$ and γ_2 a Jordan curve around zero (for details, see Introduction, Fig. 1).

From Theorem 1 above these polynomials are represented by means of the Rodrigues formula

$$Q_{2n}(x) = \frac{1}{w(x)} D^n[x^{2n+\alpha_0}(x-a)^{n+\alpha_1} e^{\gamma/x}], \tag{30}$$

which gives us, for the leading coefficients and the constant term of $Q_{2n} = k_{2n,\,2n} x^{2n} + \cdots + k_{0,\,2n}$, the expressions

$$\begin{aligned} k_{2n,\,2n} &= (\alpha_0+\alpha_1+3n)(\alpha_0+\alpha_1+3n-1)\cdots(\alpha_0+\alpha_1+2n+1) \\ k_{0,\,2n} &= (a\gamma)^n. \end{aligned} \tag{31}$$

3.1. *Generating Function for J–B Polynomials*

We use the Darboux method for the study of the asymptotics of J–B polynomials. This method is based on an analysis of the singularities on the boundary of the largest holomorphy disk of the generating function for polynomials $\{Q_{2n}\}$:

$$F(z; x) = \sum_{n=0}^{\infty} \frac{Q_{2n}(x)}{n!} \frac{1}{z^{n+1}}. \tag{32}$$

The method's main steps are:

1. the determination of the singularities for $F(z)$ in the z-plane with the largest modulus with respect to its dependence on x;
2. singularity type description and computation of the residues.

Then, applying the general theorem on the limit of the coefficients of a converging power series, we will have the asymptotics for $\{Q_{2n}\}$.

3.1.1. *A Representation for the Generating Function $F(z; x)$ by Means of an Algebraic Function*

THEOREM 3.1. *The generating function $F(z; x)$, given in (32), for the sequence of J–B polynomials $\{Q_{2n}\}$, given by (30) and (29), has the form*

$$F(z; x) = \frac{w(t_*(z))}{w(x)(z - 3t_*^2(z) + 2at_*(z))}, \tag{33}$$

where $t_*(z)$ is a branch of the algebraic function of the third order $t(z)$;

$$t^3 - at^2 - zt + zx = 0, \tag{34}$$

such that $t_*(z) \to x$, when $z \to \infty$.

Proof. We employ a standard technique. From Rodrigues' formula (30) we have by means of the Cauchy integral ($\phi(x) = x^2(x-a)$)

$$Q_{2n}(x) = \frac{1}{w(x)} D^n[\phi^n(x)\, w(x)] = \frac{n!}{w(x)} \frac{1}{2\pi i} \int_{\Gamma_x} \frac{w(t)\, \phi^n(t)}{(t-x)^{n+1}}\, dt,$$

where Γ_x is a circle with its center at x and a sufficiently small radius such that the origin and a do not belong to the disk Δ_x, bounded by $\Gamma_x (\Gamma_x = \partial(\Delta_x))$ in the counterclockwise sense. The same branch for the multivalued function $w(x)$ is considered inside and outside the integral. For $|z|$ large enough,

$$\left| \frac{\phi(t)}{t-x} \frac{1}{z} \right| < 1,$$

when t belongs to Γ_x, so the series

$$\sum_{n=0}^{\infty} \left(\frac{\phi(t)}{(t-x)\, z} \right)^n = \frac{z(t-x)}{z(t-x) - \phi(t)}$$

converges uniformly for $t \in \Gamma_x$. As a consequence

$$F(z; x) = \frac{1}{w(x)} \frac{1}{2\pi i} \int_{\Gamma_x} \frac{w(t)}{z(t-x) - t^2(t-a)}\, dt; \qquad x \neq 0, \quad x \neq a. \tag{35}$$

Since

$$z(t-x) - t^2(t-a) = 0 \tag{36}$$

if and only if

$$t = x + \frac{t^2(t-a)}{z}, \qquad z \neq 0,$$

and $x + t^2(t-a)/z$ is a contractive mapping for large enough values of $|z|$ which transforms the disk Δ_x into the same disk, then Eqs. (34) and (35) have a unique root $t_*(z)$ within Γ_x and it is a simple root. By the residue theorem, from (35) we derive (33).

Moreover, $t_*(z)$ tends to x when z tends to infinity and the other two functions, $t_1(z)$ and $t_2(z)$ defined by Eq. (34), from Vieta's formulas, tend to infinity when z tends to infinity. ▮

Remark. The algebraic function (34), which stands for $F(z; x)$ in the representation (33), is of order 3. In comparison with classical OP (see the original version of the Darboux method for Jacobi polynomials [30]), where the generating function is represented by a second-order algebraic function, the case under consideration requires a more involved analysis.

3.1.2. *Singularities of the Generating Function $F(z; x)$ in the z-Plane*

THEOREM 3.2. *The generating function $F(z; x)$, given by* (32) *and* (33), *for the sequence of J–B polynomials* (30) *and* (29), *has three singularities in the z-plane, which* (*with respect to their dependence on x*) *are*

$$\begin{aligned} z &:= z_0 = 0 \\ z &:= z_\pm(x) = \tfrac{1}{8}\left(27x^2 - 18ax - a^2 \pm (9x-a)\sqrt{(9x-a)(x-a)}\right). \end{aligned} \tag{37}$$

Proof. The singularities of the algebraic function $t_*(z)$ are the solutions of the system

$$\begin{cases} z(t-x) - t^2(t-a) = 0 \\ z - 3t^2 + 2at = 0. \end{cases} \tag{38}$$

Then

$$t(2t^2 - (3x+a)\,t + 2ax) = 0,$$

whence

$$t = 0 \qquad \text{or} \qquad t = t_\pm(x) = \tfrac{1}{4}\left(3x + a \pm \sqrt{(9x-a)(x-a)}\right). \tag{39}$$

As a consequence, using the second equation in (38), $t_*(z)$ has the singularities (37). For $z_\pm(x)$ the branch of $\sqrt{(9x-a)(x-a)}$ is defined by the condition

$$\sqrt{(9x-a)(x-a)} = a\sqrt{(9\lambda-1)(\lambda-1)}$$

for $x=\lambda a$, λ a real number, and the last square root is positive for $\lambda>1$.

Since $t_*(z)=0$ or $t_*(z)=a$ is only possible when $z=0$, but $x\neq 0$, $x\neq a$, the generating function $F(z;x)$ does not have any singularities different from $z=0$, $z=z_\pm(x)$. ∎

3.1.3. *Comparative Analysis of the Modulus for Singular Points of the Generating Function* $F(z;x)$

Now we determine, for each value of the parameter x, which singularities of $F(z;x)$ belong to the boundary of the largest disk of holomorphy of $F(z;x)$ with center at infinity (i.e., we determine which singularities have the largest modulus).

THEOREM 3.3. *Let $\mathscr{G}$ be an analytic curve in the x-plane, described by the roots of the equation*

$$\mathscr{G} := \left\{ x\colon 729\left(\frac{x}{a}\right)^4 - 972\left(\frac{x}{a}\right)^3 + 270\left(\frac{x}{a}\right)^2 + 4(1-4t)\frac{x}{a} + 1 = 0,\ -2\leqslant t\leqslant 2 \right\}, \tag{40}$$

when the parameter t varies in $[-2,2]$ *(i.e, $\mathscr{G}$ is the union of the interval* $[\frac{1}{9}a, a]$ *and a loop $\mathscr{L}$ around zero, joined with the interval at the point* $\frac{1}{9}a$ *(see Fig. 2)):*

$$\mathscr{G} = [\tfrac{1}{9}a, a] \cup \mathscr{L}.$$

Let $\Omega_\pm$ be a partition of the x-plane such that Ω_+ is a neighborhood of infinity bounded by $\mathscr{G}$ and Ω_- is a neighborhood of zero bounded by $\mathscr{L}$:

$$\begin{aligned} \partial(\Omega_+) &= \mathscr{G}, & \infty &\in \Omega_+, \\ \partial(\Omega_-) &= \mathscr{L} & 0 &\in \Omega_-. \end{aligned} \tag{41}$$

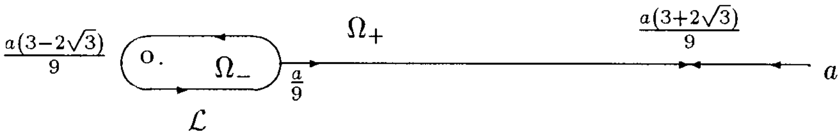

FIGURE 2

Then the moduli of the singularities (37) *of the generating function* $F(z; x)$ ((32) *and* (33)) *satisfy*

$$|z_+(x)| > |z_-(x)| > 0, \qquad x \in \Omega_+$$
$$|z_-(x)| > |z_+(x)| \geqslant 0, \qquad x \in \Omega_-$$
$$|z_+(x)| = |z_-(x)|, \qquad x \in \mathcal{G} = [\tfrac{1}{9}a, a] \cup \mathcal{L}.$$

Proof. Let us consider the following set in the x-plane

$$\{x \colon |z_+(x)| = |z_-(x)|\}. \tag{43}$$

Taking into account that (43) holds if and only if

$$\frac{z_+(x)}{z_-(x)} + \frac{z_-(x)}{z_+(x)} = t, \qquad -2 \leqslant t \leqslant 2,$$

and that $z_+(x)$ and $z_-(x)$ are the roots of the equation

$$z^2 - \frac{27x^2 - 18ax - a^2}{4} z + a^3 x = 0, \tag{44}$$

by Vieta's formula we have

$$\{x \colon |z_+(x)| = |z_-(x)|\} = \left\{x \colon \frac{z_+^2(x) + z_-^2(x)}{z_+(x)\, z_-(x)} = t,\ -2 \leqslant t \leqslant 2\right\},$$

which give us the Eq. (40) for $\mathcal{G}$.

The discriminant of this equation is $(t-2)^2\,(t+2)^2$ times a constant, so we have four functions $x_i(t)$, $i = 1, 2, 3, 4$, with branch points at $t = \pm 2$. For $t = 2$ the equation has $x(2) = a/9$ as triple root and $x(2) = a$ as a single root. For $t = -2$ we have that $x(-2) = (a/9)(3 + 2\sqrt{3})$ and $x(-2) = (a/9)(3 - 2\sqrt{3})$ are double roots. Since the Eq. (40) defining $\mathcal{G}$ has real coefficients, and taking into account that

$$|z_+(x)| = |z_-(x)|, \qquad x \in \left[\frac{a}{9}, a\right],$$

we have for $t \in [-2, 2]$

$$x_1(t) \in \left[\frac{a}{9}(3 + 2\sqrt{3}), a\right],$$

$$x_2(t) \in \left[\frac{a}{9}, \frac{a}{9}(3 + 2\sqrt{3})\right],$$

$x_3(t)$ is a Jordan arc joining $a/9$ and $(a/9)(3-2\sqrt{3})$, and $x_4(t)$ is symmetric to $x_3(t)$ with respect to the straight line defined by a and the origin.

Then, $\mathscr{G}$ is formed by the segment $[a/9, a]$ and by a loop $\mathscr{L}$ around the origin (see Fig. 2).

Let $\Omega_{\pm}$ be the domains defined in the statement of the theorem. Since

$$\lim_{x\to\infty} z_+(x)=\infty,$$

$$\lim_{x\to\infty} z_-(x)=0,$$

$$z_+(0)=0,$$

$$z_-(0)\neq 0, \qquad x\in\mathbf{C},$$

it follows that (42) holds. ▮

3.1.4. *An Explicit Representation for the Generating Function $F(z; x)$ with the Help of Cardano's Formula*

The representation (33) for $F(z; x)$ given by Theorem 3.1 can be modified in the following form.

THEOREM 3.4. *The generating function $F(z; x)$, given in* (32) *and* (33) *for the sequence of J–B polynomials* (30) *and* (29), *has the form*

$$F(z; x)=\frac{w(t_*(z))}{w(x)}\frac{\sqrt{27}}{6i}\frac{d_*(z)-g_*(z)}{z\sqrt{(1-(z_+(x))/z)(1-(z_-(x))/z)}}, \tag{45}$$

where $z_{\pm}$ are the non-zero singularities of $F(z; x)$ given in (37), *$t_*(z)$ is the same as in* (33), *and $d_*(z)$ and $g_*(z)$ are the branches of the functions*

$$\begin{aligned} g(z)&=\left(\frac{2a^3+(9a-27x)z}{54z\sqrt{z}}+\frac{i}{\sqrt{27}}\sqrt{\left(1-\frac{z_-(x)}{z}\right)\left(1-\frac{z_+(x)}{z}\right)}\right)^{1/3}, \\ d(z)&=\left(\frac{2a^3+(9a-27x)z}{54z\sqrt{z}}-\frac{i}{\sqrt{27}}\sqrt{\left(1-\frac{z_-(x)}{z}\right)\left(1-\frac{z_+(x)}{z}\right)}\right)^{1/3}, \end{aligned} \tag{46}$$

such that

$$\lim_{z\to\infty} g_*(z)=-\frac{i}{\sqrt{3}}, \qquad \lim_{z\to\infty} d_*(z)=\frac{i}{\sqrt{3}}. \tag{47}$$

Proof. The discriminant of Eq. (34)

$$t^3 - at^2 - zt + xz = 0$$

from the representation of $F(z; x)$ (33) is

$$\begin{aligned} D &= 4z\left(z^2 - \frac{27x^2 - 18ax - a^2}{4}z + a^3x\right) \\ &= 4z(z - z_-(x))(z - z_+(x)), \end{aligned}$$

and, by Cardano's formula, the solutions are

$$\begin{aligned} t_k(z) - \frac{a}{3} &= \sqrt{z}\left(g_k(z) + \frac{a^2 + 3z}{9zg_k(z)}\right) \\ &= \sqrt{z}\left(d_k(z) + \frac{a^2 + 3z}{9zd_k(z)}\right) = \sqrt{z}(g_k(z) + d_k(z)), \end{aligned}$$

where $g_k(z)$ and $d_k(z)$, for $k = 1, 2, *$, are the different cubic roots of (46).

We also have

$$\lim_{z \to \infty} g_k(z) = \left(\frac{i}{\sqrt{27}}\right)^{1/3}, \qquad \lim_{z \to \infty} d_k(z) = \left(\frac{-i}{\sqrt{27}}\right)^{1/3},$$

and since $\lim_{z \to \infty} t_*(z) = x$ (see (34))

$$t_*(z) = \frac{a}{3} + \sqrt{z}\,(g_*(z) + d_*(z)), \tag{48}$$

where the branches d_* and g_* satisfy (47) and the branch for $\sqrt{z}$ is chosen with the only condition that the outside and inside brackets are equal.

From (48) we obtain (45). ▮

3.1.5. *Type of Singularities of the Generating Function* $F(z; x)$

THEOREM 3.5. *The singularities* $z_+(x)$, $z_-(x)$, *and* (37) *of the generating function* $F(z; x)$, *given in* (32), (33), *and* (45), *are branch points of the second order. In punctured neighborhoods* $\varepsilon(z_\pm)$ *of these points,* $F(z; x)$ *has the form*

$$\begin{aligned} F(z; x) &= \frac{A(x)}{\sqrt{1 - (z_+(x))/z}} + o(1), \qquad z \in \varepsilon(z_+), \\ F(z; x) &= \frac{B(x)}{\sqrt{1 - (z_+(x))/z}} + o(1), \qquad z \in \varepsilon(z_-), \end{aligned} \tag{49}$$

where $x \notin \{0; a; a/9\}$ *and*

$$A(x) = \frac{w(t_+(x))}{w(x)} \frac{d_*(z_+) - g_*(z_+)}{z_+ \sqrt{1-(z_-(x))/(z_+(x))}} \frac{\sqrt{3}}{2i},$$
$$B(x) = \frac{w(t_-(x))}{w(x)} \frac{d_*(z_-) - g_*(z_-)}{z_- \sqrt{1-(z_+(x))/(z_-(x))}} \frac{\sqrt{3}}{2i}, \tag{50}$$

where $t_\pm(x)$ *is defined by* (39) *and* g_* *and* d_* *by* (46) *and* (47).

Proof. We assume x to belong to the line $x = \delta a$, $\delta > 1$. Denoting

$$D^* = \left(1 - \frac{z_+(x)}{z}\right)\left(1 - \frac{z_-(x)}{z}\right),$$

when z moves around $z_+(x)$, after one circuit, the function $\sqrt{D^*}$ changes to $-\sqrt{D^*}$. We describe the situation by $\sqrt{D^*} \to -\sqrt{D^*}$. Then

$$g_*(z) \to c d_*(z),$$
$$d_*(z) \to c^2 g_*(z),$$

where c is a cubic root of the unity. We now determine the constant c.

Let us consider the curve J^ε consisting of the line $(1/\lambda)\, z_+(x)$, $0 < \lambda < 1 - \varepsilon$, for some $\varepsilon > 0$, the circle

$$|z - z_+(x)| = \frac{\varepsilon}{1-\varepsilon} |z_+(x)|$$

traversed counterclockwise, and the line once more in the opposite direction. When z goes through $J^\varepsilon(x)$, the function $g_*^3(z)$ describes a Jordan arc, $C^\varepsilon(x)$, beginning at $i/\sqrt{27}$ and ending at $-i/\sqrt{27}$.

Let $C(x)$ be the limit curve when ε tends to zero. For $x = \delta a$, $\delta > 1$, the curve $C(x)$ lies in the half-plane $\Re(z) < 0$ (we assume $\sqrt{z}$ in the definition of $g_*^3(z)$ such that $\sqrt{a^2 \xi} = a\sqrt{\xi}$ and $\sqrt{\xi} > 0$ for $\xi > 0$) and the argument of $g^3(z)$ increases by $+\pi$. Hence the new value of $g_*(z)$ at infinity is $(-i/\sqrt{3})\, e^{(\pi/3)i}$. So

$$\frac{-i}{\sqrt{3}} e^{(\pi/3)\,i} = c d_*(\infty) = c \frac{i}{\sqrt{3}},$$

which means that

$$c = -e^{(\pi/3)\,i}.$$

Then, after a circuit around $z_+(x)$, $F(z; x)$ changes its value and after a second circuit it returns to the original value.

For $x=\delta a$, $\delta<(3-2\sqrt{3})/9$ (x is outside of the loop), the corresponding value of c is $-e^{(-\pi/3)\,i}$ which means that $z_+(x)$ is a second order branch point of $F(z; x)$.

For $x=\delta a$, $(3-2\sqrt{3})/9<\delta<\frac{1}{9}$, $z_-(x)$ is not a singular point because $z=-a^2/3$ is a triple zero of $g^3(z)$ and after a circuit from infinity to infinity around $z_-(x)$, the argument of $g^3(z)$ is increased by 3π and this means that

$$g_*(z)\to d_*(z),$$
$$d_*(z)\to g_*(z).$$

Moving x through the curves $\{x\colon |z_\pm(x)|=const.\}$ and using the continuity in x of the function

$$\frac{1}{2\pi i}\int_\gamma \frac{(g^3(z))'}{g^3(z)}\,dz$$

for appropriate curves γ from infinity to infinity, we obtain from the behaviour of $z_\pm(x)$ when $x=\delta a$ that $z_+(x)$ is a branch point of $t_*(z)$ when x belongs to Ω_+ and $z_-(x)$ is not a branch point when x belongs to $\Omega_-\cup\mathscr{L}\backslash\{0\}$. In this domain, $z_-(x)$ is a branch point for the two other branches of the third-order algebraic function $t(z)$. Then $z_+(x)$ is the singularity on the largest disk of holomorphy of $F(z; x)$ when x belongs to $\mathbf{C}\backslash([a/9, a]\cup\{0\})$.

When x belongs to $(a/9, a)$, taking into account that $z_\pm(x)$ are singular points of $t_*(z)$ if and only if $\lim_{z\to z_\pm} t_*(z)=t_\pm(x)$ ($t_\pm(x)$ is defined by (39)), by the Schwarz reflection principle, we have that $z_+(x)$ is a singular point if and only if $z_-(x)$ is a singular point. So $z_+(x)$ and $z_-(x)$ are branch points; otherwise $t_*(z)$ would be a bounded holomorphic function on $\mathbf{C}\backslash\{0\}$. Moreover, when $z_\pm(x)$ is a singular point,

$$\lim_{z\to z_\pm}(g_*(z)-d_*(z))\neq 0,$$

which leads to (49). The expressions (50) for $A(x)$ and $B(x)$ are obtained by taking the limits

$$A(x)=\lim_{z\to z_+}\sqrt{1-\frac{z_+(x)}{z}}\,F(z; x),$$

$$B(x)=\lim_{z\to z_-}\sqrt{1-\frac{z_-(x)}{z}}\,F(z; x). \quad \blacksquare$$

3.2. *Strong Asymptotics for J–B Polynomials*

We use the following well-known result for the limit of the Taylor coefficients (Darboux's method)

Let

$$f(z)=\sum_{n=0}^{\infty} a_n z^n$$

be holomorphic in the unit disk, which is its maximal disk of holomorphy. Let there be k singular points of $f(z)$ on the unit circle, $1/w_j$, $j=1, ..., k$, and let these all be algebraic points. In a punctured neighborhood of each point $1/w_j$, let f have an expansion of the form

$$f(z)=b_0^{(j)}(1-w_j z)^{m_j/p_j}+b_1^{(j)}(1-w_j z)^{(m_j+1)/p_j}+\cdots,$$

where $m_j \in \mathbf{Z}$, $p_j \in \mathbf{N}$, $m_j/p_j \notin \mathbf{Z}_+$, $b_0^{(j)} \neq 0$, and the expansion is carried out in powers of $(1-w_j z)^{1/p_j}$. Then (as $n \to \infty$)

$$a_n=\sum_{j=1}^{k} b_0^{(j)} \binom{-m_j/p_j+n-1}{n} w_j^n(1+O(n^{-1/p_j})). \tag{51}$$

If a_n depends on a parameter x, then (51) holds uniformly in an appropriate domain of variation of x.

(For the proof see [32, p. 206, 8.4].)

From the properties of the generating function $F(z; x)$, studied in the previous subsection (Theorems 3.2, 3.3, 3.5), we have

THEOREM 3.6. *For J–B polynomials* (30) *and* (29), *the following asymptotic formulae hold* (*as* $n \to \infty$):

1. $$\frac{1}{n!} Q_{2n}(x)=\frac{A(x)}{\sqrt{2\pi n}}(z_+(x))^{n+1}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right), \tag{52}$$

uniformly with respect to x on compact subsets of $\mathbf{C}\backslash([a/9, a] \cup \{0\})$.

2.

$$\frac{1}{n!} Q_{2n}(x)=\frac{1}{\sqrt{2\pi n}}\{A(x)(z_+(x))^{n+1}+B(x)(z_-(x))^{n+1}\}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right), \tag{53}$$

uniformly with respect to x on compact subsets of $(a/9, a)$.

Here $z_{\pm}(x)$ *are as defined in* (37) *and* $A(x)$ *and* $B(x)$ *are as in* (50).

In the following two corollaries, we derive the asymptotic formula for $\{Q_{2n}\}$ when x belongs to the interval $[a/9, a]$.

COROLLARY 3.1. *Let* α_0, α_1, *and* γ/a *be real numbers. Then for J–B polynomials* (30) *and* (29) *the following asymptotic formula holds* (*as* $n\to\infty$)

$$\frac{Q_{2n}(x)}{n!}=\frac{\sqrt{27}\,a^{2n}}{6i\sqrt{2\pi n}}\left\{W(\lambda)\,\frac{d_*(a^2z(\lambda))-g_*(a^2z(\lambda))}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}}\,z^n(\lambda)\right.$$
$$\left.-\overline{W(\lambda)}\,\overline{\frac{d_*(a^2z(\lambda))-g_*(a^2z(\lambda))}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}}}\,\overline{z^n(\lambda)}\right\}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right), \quad (54)$$

where $x=\lambda a$, *uniformly with respect to* λ *on compact subsets of the interval* $(1/9, 1)$, *where* g_* *and* d_* *are defined in* (46) *and* (47); $z(\lambda)=z_+(x)/a^2$, *and* $W(\lambda)=w(t_*(z_+(x)))/w(x)$.

Proof. Setting $x=\lambda a$, $1/9<\lambda<1$, we have from (37)

$$z_+(x)=\frac{a^2}{8}\,(27\lambda^2-18\lambda-1+i(9\lambda-1)\sqrt{(9\lambda-1)(1-\lambda)})=:a^2z(\lambda), \qquad (55)$$

and $z_-(x)=a^2\,\overline{z(\lambda)}$. On the other hand, as it was shown in (39)

$$t_*(z_+(x))=\frac{1}{4}\,(3x+a+\sqrt{(9x-a)(x-a)})$$
$$=\frac{a}{4}\,(3\lambda+1+i\sqrt{(9\lambda-1)(1-\lambda)})=:at(\lambda), \qquad (56)$$

and $t_*(z_-(x))=a\overline{t(\lambda)}$. Moreover, for $z=a^2\xi$, $\xi>1$, we have

$$g^3(a^2\xi)=\frac{2+(9-27\lambda)\,\xi}{54\xi\sqrt{\xi}}+\frac{i}{\sqrt{27}}\sqrt{\left|1-\frac{z(\lambda)}{\xi}\right|^2},$$

so, $d^3(a^2\xi)=\overline{g^3(a^2\xi)}$ which means that $d(a^2\xi)=c\overline{g(a^2\xi)}$, where c is a cubic root of unity. Taking into account that

$$\lim_{\xi\to\infty}g(a^2\xi)=-\frac{i}{\sqrt{3}}, \qquad \lim_{\xi\to\infty}d(a^2\xi)=\frac{i}{\sqrt{3}},$$

then $c=1$ and $d(a^2\xi)=\overline{g(a^2\xi)}$ follows.

Thus $d(a^2\xi) - g(a^2\xi) = 2i\, \Im g(a^2\xi)$, and by the Schwarz reflection principle

$$d(z_-(x)) - g(z_-(x)) = -\overline{d(z_+(x)) - g(z_+(x))}, \qquad x \in \left[\frac{a}{9}, a\right], \tag{57}$$

because $z_+(x)$ and $z_-(x)$ are symmetric points with respect to the straight line $z = a^2\xi$, $\xi > 1$. Finally, it is clear that

$$\sqrt{1 - \frac{z_+(x)}{z_-(x)}} = \overline{\sqrt{1 - \frac{z_-(x)}{z_+(x)}}}, \qquad x \in \left[\frac{a}{9}, a\right],$$

and by (56) and by the definition of $w(x)$ (29),

$$W(\lambda) := \frac{w(t_*(z_+(x)))}{w(x)} = \left(\frac{t(\lambda)}{\lambda}\right)^{\alpha_0} \left(\frac{1 - t(\lambda)}{1 - \lambda}\right)^{\alpha_1} \exp\left[\frac{\gamma}{a}\left(\frac{\overline{t(\lambda)} - 1}{\lambda}\right)\right], \tag{58}$$

where $x^{\alpha_0}(1-x)^{\alpha_1} = |x|^{\alpha_0}\, |1-x|^{\alpha_1}$ for $0 < x < 1$. Then, for α_0 and α_1 real numbers and γ such that γ/a is real too, we obtain

$$\frac{w(t_*(z_-(x)))}{w(x)} = \overline{W(\lambda)}.$$

Hence, for α_0, α_1, and γ/a real numbers, (55), (56), (57), and (58) lead to (54).

Setting

$$D(\lambda) =: W(\lambda)\, \frac{d_*(a^2 z(\lambda)) - g_*(a^2 z(\lambda))}{\sqrt{1 - \overline{z(\lambda)}/z(\lambda)}}, \tag{59}$$

the expression (51) in Corollary 3.1 can be rewritten as follows:

COROLLARY 3.2. *With the assumptions and the conditions of Corollary* 3.1 *we have*

$$\frac{Q_{2n}(x)}{n!} = \frac{\sqrt{3}\, a^{2n}}{\sqrt{2\pi n}}\, |z(\lambda)|^n\, |D(\lambda)| \sin[n \arg[z(\lambda)] + \arg[D(\lambda)]] \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right), \tag{60}$$

$x = \lambda a$, *uniformly with respect to* λ *on compact subsets of* $(1/9, 1)$, *where*

$$\arg[z(\lambda)] = \pi - \frac{1}{2}\int_{1/9}^{\lambda} \frac{9t - 1}{t\sqrt{(9t-1)(1-t)}}\, dt,$$

and

$$\arg[D(\lambda)] = -\frac{1}{4}\int_{1/9}^{\lambda}\left(6(\alpha_0+\alpha_1+1)\,t^2 - \left(2\alpha_0+5\,\frac{\gamma}{a}\right)t\right)\frac{dt}{t^2\sqrt{(9t-1)(1-t)}} + A, \qquad (61)$$

and A is a constant different from π and $-\pi$.

Proof. The representation (60) immediately follows from the definition (59) and (54). Concerning the expression for $\arg D[\lambda]$, after some calculation we obtain

$$\arg D(\lambda) = \frac{\alpha_0}{2}\int_{1/9}^{\lambda}\frac{1-3t}{t\sqrt{(9t-1)(1-t)}}\,dt - \frac{3}{2}\alpha_1\int_{1/9}^{\lambda}\frac{dt}{\sqrt{(9t-1)(1-t)}} - \frac{\gamma}{a}\int_{1/9}^{\lambda}\frac{1-5t}{4t^2\sqrt{(9t-1)(1-t)}}\,dt + \arg\frac{d_*(a^2z(\lambda))-g_*(a^2z(\lambda))}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}}.$$

Moreover,

$$\frac{d_*(a^2z(\lambda))-g_*(a^2z(\lambda))}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}} = \frac{d_*(a^2z(\lambda))(1-c)}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}} = \frac{(2+(9-27\lambda)\,z(\lambda))^{1/3}\,(1-c)}{\sqrt{2i\sqrt{(9\lambda-1)^3\,(1-\lambda)}}},$$

for some cubic root of $2+(9-27\lambda)\,z(\lambda)$, where c is a cubic root of unity different from 1. From this expression we obtain

$$\arg\frac{d_*(a^2z(\lambda))-g_*(a^2z(\lambda))}{\sqrt{1-\overline{z(\lambda)}/z(\lambda)}} = A - \frac{3}{2}\int_{1/9}^{\lambda}\frac{dt}{\sqrt{(9t-1)(t-1)}},$$

where

$$A = \lim_{\lambda\to 1/9}\arg(2+(9-27\lambda)\,z(\lambda))^{1/3} - \arg\sqrt{i} + \arg(1-c),$$

which, as it is easily seen, is different from π and $-\pi$ for any root and for any possible value of c. As a consequence we have (61). ∎

3.3. *Asymptotics of the Zeros of J–B Polynomials*

We can apply the formulae of the strong asymptotics (see Theorem 3.7 and corollaries) to locate the zeros $\{x_{j,2n}\}_{j=1}^{2n}$ of the polynomials Q_{2n}, as $n\to\infty$.

First, we count the number of zeros on the interval $[a/9, a]$, where the polynomials oscillate.

THEOREM 3.7. *Let the exponents of the J–B weight function* (29), $w(x)$, α_0, α_1, *and* γ/a *be real numbers. If* k_0 *is an integer such that*

$$(k_0 - 1)\,\pi < \arg[D(\lambda)] < k_0\pi, \tag{62}$$

where $\arg[D(\lambda)]$ *is given in* (61), *then for* n *large enough, the J–B polynomial* Q_{2n}, (30), *has* $n - k_0$ *or* $n - k_0 + 1$ *zeros on the segment* $(a/9, a)$.

Proof. From the asymptotic formula (60) we have that Q_{2n} has a zero on $(a/9, a)$ if and only if

$$H_{n,k}(\lambda) := \arg z(\lambda) + \frac{\arg D(\lambda) - k\pi}{n}$$

has a zero on $(1/9, 1)$ for some integer number k. Moreover, for n large enough, $H_{n,k}(\lambda)$ is a monotonic function.

Since $H_{n,k}(1/9) = ((n-k)\,\pi + A)/n$ and $(k_0 - k - 1)\,\pi/n < H_{n,k}(1) < (k_0 - k)\,\pi/n$, from (62) it is obvious that

$$k < k_0 \Rightarrow H_{n,k}(1/9) > 0,$$

$$H_{n,k}(1) > 0 \Rightarrow H_{n,k}(\lambda) \neq 0, \qquad 1/9 < \lambda < 1.$$

$$k_0 < k < n \Rightarrow H_{n,k}(1/9) > (A + \pi)/n > 0, \qquad H_{n,k}(1) < 0;$$

thus $H_{n,k}(\lambda)$ has exactly one zero for $1/9 < \lambda < 1$:

$$k > n \Rightarrow H_{n,k}(1/9) < (A - \pi)/n < 0,$$

$$H_{n,k}(1) < 0 \Rightarrow H_{n,k}(\lambda) \neq 0, \qquad 1/9 < \lambda < 1.$$

Hence, $Q_{2n}(x)$ has $n - k_0$ or $n - k_0 + 1$ zeros on the segment $[a/9, a]$.

Thus, about half of the zeros of the polynomials $Q_{2n}(x)$ are located on the interval $[a/9, a]$. "On the average" the rest of the zeros tend to the origin. Indeed, comparing the leading coefficient and the constant term of the polynomials $Q_{2n}(x)$, (31), we have

$$\prod_{j=1}^{2n} x_{j,2n} = \frac{(a\gamma)^n}{(\alpha_0 + \alpha_1 + 3n)(\alpha_0 + \alpha_1 + 3n - 1) \cdots (\alpha_0 + \alpha_1 + 2n + 1)}.$$

As far as we know, $n-k_0$ zeros lie on the interval $[a/9, a]$ (where the integer k_0 does not depend on n), i.e.,

$$\left|\frac{a}{9}\right|^{n-k_0} \leqslant \prod_{j=1}^{n-k_0} |x_{j,\,2n}| \leqslant |a|^{n-k_0};$$

therefore, the geometric mean of the rest of the zeros tends to the origin with a rate $O(1/n)$

$$\left(\prod_{j=n-k_0+1}^{2n} |x_{j,\,2n}|\right)^{1/n} \asymp \frac{1}{n}.$$

More precisely, $n+k_1(\alpha, \beta, \gamma)$ zeros tend to the origin because $z_+(x)$ has a simple zero at the origin and

$$\frac{1}{2\pi i}\int_{|x|=\varepsilon} \frac{(Q_{2n}(x)/z_+^n(x))'}{Q_{2n}(x)/z_+^n(x)}\,dx = \frac{1}{2\pi i}\int_{|x|=\varepsilon} \frac{A'(x)}{A(x)}\,dx = k_1(\alpha, \beta, \gamma),$$

for large enough values of n.

Since the main term of the asymptotics for $Q_{2n}(x)$ is known (i.e., the algebraic function $z(x)$, (44)), the potential theoretic description of the limit of the zero distribution can be obtained: Let

$$\nu_{2n} := \nu[Q_{2n}] := \frac{1}{n}\sum_{j=1}^{2n} \delta(x - x_{j,\,2n}) \tag{63}$$

be the so-called zero counting measure of the polynomial

$$Q_{2n}(x) = k_{2n;\,2n}\, q_{2n}(x) = k_{2n,\,2n} \prod_{j=1}^{2n} (x - x_{j,\,2n}).$$

We denote the logarithmic potential of the measure μ by

$$V_\mu = \int \ln \frac{1}{|x-t|}\, d\mu(t).$$

From (63), it follows that

$$V_{\nu_{2n}}(x) = -\frac{1}{n}\ln\left|\frac{Q_{2n}(x)}{k_{2n,\,2n}}\right|. \tag{64}$$

Theorem 3.6 gives us

$$\frac{1}{n}\ln|Q_{2n}(x)| = \ln|z_\pm(x)| + \frac{\ln n!}{n} + o(1), \tag{65}$$

uniformly for x belonging to compact sets of $(a/9, a)$ or $\mathbf{C} \setminus ([a/9, a] \cup \{0\})$. Combining (31) and (64) with (65) we have

$$V_{\nu_{2n}}(x) = -\ln |z_{\pm}(x)| + \ln \tfrac{27}{4} + o(1). \tag{66}$$

If $x \in [a/9, a]$, then $|z_{+}(x)| = |z_{-}(x)|$ and from the algebraic equation for $z(x)$, (44), it follows that

$$|z(x)|^2 = |z_{+}(x)\, z_{-}(x)| = |a^3 x|.$$

Therefore, for $x \in (a/9, a)$, we have from (66)

$$2V_{\nu_{2n}}(x) = \ln \frac{1}{|x|} + \ln \frac{27^2}{16\, |a^3|} + o(1). \tag{67}$$

Next, we split the zero counting measure into two parts corresponding to the zeros falling on the interval $(a/9, a)$ and those outside of it:

$$\nu_{n,1} := \frac{1}{n} \sum_{j=1}^{n-k_0} \delta(x - x_{j,2n}),$$

and

$$\nu_{n,2} := \frac{1}{n} \sum_{n-k_0+1}^{2n} \delta(x - x_{j,2n}).$$

So, Eq. (67) can be rewritten as

$$2V_{\nu_{n,1}}(x) + 2V_{\nu_{n,2}}(x) - \ln \frac{1}{|x|} = \ln \frac{27^2}{16\, |a^3|} + o(1), \qquad x \in \left(\frac{a}{9}, a\right).$$

Since $n + k_1$ zeros corresponding to $\nu_{n,2}$ tend to the origin, the measure of their distribution (zero counting measure) converges weakly to the Dirac measure of mass one concentrated at the origin and

$$V_{\nu_{n,2}}(x) \to \ln \frac{1}{|x|}, \qquad n \to \infty.$$

Therefore

$$2V_{n,1}(x) + \ln \frac{1}{|x|} = \ln \frac{27^2}{16|a^3|} + o(1), \qquad x \in \left(\frac{a}{9}, a\right).$$

This means that the zero counting measure, on the interval $[a/9, a]$, converges to the equilibrium measure of the logarithmic potential in the field of a unit mass located at the origin.

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